TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 216, 1976

A PRODUCT FORMULA FOR GENERALIZATIONS OF THE KERVAIRE INVARIANT

BY

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ABSTRACT. Formulas are developed for the Arf invariant of the product of two manifolds in terms of invariants of the factors. If the Wu orientations are carefully chosen the formula is $\sigma(M \times N) = \sigma(M)\sigma(N)$.

1. **Introduction.** Recall, if M is a smooth m-manifold, $f: M \longrightarrow BO$ is the classifying map of its normal bundle and $v_i \in H^i(BO; Z/2)$ is the ith Wu class, $f_* v_i = 0$ for $2i > m$. Thus if we form a fibration over BO by killing v_i, $2i > m$, f lifts to the total space of this fibration. We call such a lift a trivialization of v_i on M.

In [4] we introduced an invariant $\sigma(M) \in Z/8$ $(Z/l = Z/lZ)$ for closed, compact smooth $2n$-manifolds M with a trivialization of v_{n+1}. Let $\sigma(M) = 0$ if dim M is odd. σ generalizes various formulations of the Kervaire invariant ([6], [3], and [1]) and it plays a role analogous to the Hirzebruch index for surgery problems involving manifolds of dimension $2n$ where n is odd. In this paper we give formulas relating $\sigma(M \times N)$, $\sigma(M)$ and $\sigma(N)$. In [2], W. Browder gave a special case of our formulas ((4.7) below). Also, D. Sullivan's product formula for surgery obstructions follows from our results ((4.8)). The results of this paper were announced in [5].

To deal with $\sigma(M \times N)$ we need a product procedure for handling the v_{n+1} trivializations. In §2 we define a Wu orientation of M to be (essentially) a trivialization of each of the classes v_i for $2i > \dim M$ and we define the product of two Wu orientations as in [2]. In the remainder of this section we assume all manifolds have Wu orientations and $M \times N$ denotes the cartesian product with the product orientation.

By analogy with the index, one might hope that $\sigma(M \times N) = \sigma(M)\sigma(N)$. Although this holds if the Wu orientations are carefully chosen (see below), it does not hold in general. The following is an interesting example of when this formula fails. $\sigma(M)$ is an algebraic invariant associated to a quadratic function $\varphi: H^n(M) \longrightarrow Z/4$ where $\dim M = 2n$. (All cohomology is with $Z/2$ coefficient.) S^m has a nontrivial Wu orientation V. Let $\bar{S}^m = (S^m, V)$. $\sigma(\bar{S}^m \times \bar{S}^m) = 4$ while $\sigma(\bar{S}^m) = 0$ since $\bar{S}^n$ has no middle dimensional cohomology.

To get around this difficulty we extend φ, by a homotopy construct, to $\varphi: H^*(M) \longrightarrow \Lambda$ where Λ is a rather complicated ring (see §3) containing $Z/4$. φ has the property that

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Received by the editors November 18, 1974.

AMS (MOS) subject classifications (1970). Primary 57B10, 57D65, 57D90.

(1)This work was carried out while the author was supported by NSF grant GP 21510 and by an SRC Fellowship at University College, London.

$$\varphi(u \otimes v) = \varphi(u)\varphi(v) \tag{1.1}$$

where $u \otimes v \in H^*(M \times N)$. We then proceed by algebra, as in [4], to associate to φ an invariant $\Sigma(\varphi)$ essentially lying in the Grothendieck ring of such gadgets. We then define $\Sigma(M)$ to be $\Sigma(\varphi)$. Σ satisfies $\Sigma(M \times N) = \Sigma(M)\Sigma(N)$. Furthermore, $\Sigma(M)$ is determined by $\sigma(M)$ and $\sigma_k(M) \in Z/2$, $k = 1, 2, \ldots$, where σ_k is characterized by $4\sigma_k(M) = \sigma(\bar{S}^k \times M)$ where $4\colon Z/2 \longrightarrow Z/8$ is the inclusion. In terms of σ and σ_k our product formulas are

$$\sigma(M \times N) = \sigma(M)\sigma(N) + 4\left(\sum \sigma_k(M)\sigma_k(n)\right),$$
$$\sigma_k(M \times N) = \sigma(M)\sigma_k(N) + \sigma_k(M)\sigma(N).$$

In defining product Wu orientations, one chooses liftings of certain maps on universal examples. Furthermore, as in [4], one chooses certain homomorphisms, in the language of [4], $h_n\colon \pi_{2n+k}(Y_k \wedge K_n) \longrightarrow Z/4$. In our present treatment, we again choose the homomorphism h_n and these choices, together with (1.1), determine the above liftings. Having made these choices and thus defined φ for all manifolds, we define a *preferred Wu orientation* of M to be a Wu orientation such that $\varphi(H^p(M)) = 0$ for $2p < \dim M$. By (1.1) the product of preferred orientations is preferred. Every manifold has a preferred orientation unique up to a trivialization of v_{n+1}, $n = [\dim M/2]$ and $\sigma_k(M) = 0$ for M with a preferred orientation. Thus for such orientations $\sigma(M \times N) = \sigma(M)\sigma(N)$.

This paper is organized as follows: §2 deals with Wu orientations and their products. In §3, $\varphi\colon H^*(M) \longrightarrow \Lambda$ is constructed, modulo the main work of this paper which consists in computing pairings between certain homotopy groups. These later results are developed in §§5, 6, 7 and 8. In §4, $\Sigma(M)$ is defined and our results are stated and proved.

Throughout we use smooth manifolds. Our results apply equally well to Poincaré complexes. To formulate the results in terms of Poincaré complexes one replaces smooth manifolds by triples (X, η, α) where X is a Poincaré complex, η is its Spivak normal bundle and $\alpha \in \pi_*(T(\eta))$ is a normal invariant.

All homology and cohomology will be with $Z/2$ coefficients. K_n denotes $K(Z/2, n)$. $l\colon Z/k \longrightarrow Z/kl$ denotes the inclusion and $l(n)$ is denoted by $l \cdot n$. Although we formulate many of our constructions in terms of spectra, we make no very serious use of spectra theory. The main point of this is to reduce the number of indices we have to carry around. Spectra in the sense of [8] is quite adequate for our purposes. All manifolds will be closed, compact and smooth. We will be talking endlessly about $u \in H^p(M)$ and $v \in H^q(N)$. m and n will denote the dimensions of M and N respectively and p and q the dimensions of u and v.

2. **Wu Orientations.** In this section we essentially reproduce the material on Wu orientations in [2]. Suppose $pt = B_0 \subset B_1 \subset \cdots \subset B_k \subset \cdots \subset B$ is one of the families of classifying spaces BO_k, BSO_k, BPL_k, BG_k, etc., and suppose ζ_k is the universal bundle over B_k. The essential feature that we require is that Whitney sums make sense. Let $p\colon B^m \longrightarrow B$ be the fibration induced from the contractible fibration

by the map

$$\prod_{2i>m} v_i\colon B \longrightarrow \prod_{2i>m} K_i$$

where $v_i \in H^i(B)$ is the Wu class. We define $p_m\colon B_k^m \longrightarrow B_k$ and ζ_k^m to be the pull back of p by the inclusion $B_k \subset B$ and $\zeta_k^m = p_m * \zeta_k$, respectively. Recall, under the Whitney sum map

$$B_k \times B_l \longrightarrow B_{k+l}, \qquad v_i \longrightarrow \sum v_j \otimes v_{i-j}.$$

Hence the Whitney sum maps lift to maps $\bar{\mu}\colon B_k^m \times B_l^n \longrightarrow B_{k+l}^{m+n}$ which in turn give maps $\mu\colon \zeta_k^m \times \zeta_k^n \longrightarrow \zeta_k^{m+n}$. Let TB^m be the Thom spectra $\{T(\zeta_k^m)\}$, $T =$ Thom space. With a little effort one can choose the maps μ consistently with respect to k and l so as to give maps $\mu\colon TB^m \wedge TB^n \longrightarrow TB^{m+n}$. We call the collection, $\{\zeta_k^m, \mu\}$ a B-Wu system. Note that the maps μ involve the choice of liftings $\bar{\mu}$.

Suppose M is a manifold. We assume our manifolds come with smooth embeddings, $i\colon M \subset R^{m+k}$, k large. Let $\nu = \nu_M$ be the normal bundle of i and let $T(\nu)$ denote the Thom spectrum whose lth term is $T(\nu + O^{l-k})$. The Thom-Pontrjagin construction yields an element $\alpha_M \in \pi_m(T(\nu_M))$.

Definition (2.1). A *B-Wu orientation* of an m-manifold M is a bundle map $V\colon \nu_M \longrightarrow \zeta_k^m$. Two such orientations are *equivalent* if they are homotopic. If U and V are orientations of M and N, respectively, $U \times V$ is the orientation on $M \times N$ given by

$$\nu_{M\times N} = \nu_M \times \nu_N \xrightarrow{U\times V} \zeta_k^m \times \zeta_l^n \xrightarrow{\mu} \zeta_{k+l}^{m+n}.$$

Since v_i is zero on M for $2i > m$, we have

Lemma (2.2). *If ν_M has a B-structure, M has a B-Wu orientation.*

The standard trivialization of the normal bundle of S^m gives a map $V\colon \nu_{S^m} \longrightarrow \zeta_k$ over the constant map of S^m into B_k. Let $g\colon S^m \longrightarrow B_k^m$ represent the element of $\pi_m(B_k^m)$ coming from K_m in the fibre of $B_k^m \longrightarrow B_k$. g and V give a B-Wu orientation U of S^m. Let $\bar{S}^m = (S^m, U)$.

Remark. When $n = 1, 3$ or 7, $g\colon S^m \longrightarrow B_k^m$ is homotopic to the constant map, but under this homotopy, U is transformed into the nontrivial framing of ν_{S^m}. (See proof of (4.5).)

Remark (2.3). An alternate and technically somewhat simpler approach to Wu orientations may be done as follows: Let Y^m be the spectrum whose lth term is the two stage Postnikov system with k-invariant $\Pi_{2i>m}\chi(\mathrm{Sq}^i)\colon K_l \longrightarrow \Pi K_{l+i}$; χ is the canonical antiautomorphism of the Steenrod algebra. The cup product maps $K_{l_1} \wedge K_{l_2} \longrightarrow K_{l_1+l_2}$ lift to maps $\mu\colon \mathrm{Y}^m \wedge \mathrm{Y}^n \longrightarrow \mathrm{Y}^{m+n}$. A $Y = \{\mathrm{Y}^m, \mu\}$ orientation of an m-manifold is a map $V\colon T(\nu_M) \longrightarrow \mathrm{Y}^m$ such that the generator of $H^0(\mathrm{Y}^m)$ pulls back to the Thom class. Product orientations are defined as above and everything in this paper can be done with Y^m replacing TB^m.

3. **The main geometric construction.** Suppose M is a manifold, $T(\nu_M)$ is its normal bundle Thom spectrum, $\alpha_M \in \pi_m(T(\nu_M))$ is the Thom Pontrjagin element and

TB^m are all as in §2. Let $V\colon T(\nu_M) \longrightarrow TB^m$ come from a B-Wu orientation for M. Let $G_p^m = \pi_m(TB^m \wedge K_p)$. The cup product map $c\colon K_p \wedge K_q \longrightarrow K_{p+q}$ and $\mu\colon TB^m \wedge TB^n \longrightarrow TB^{m+n}$ define a pairing $G_p^m \otimes G_q^n \longrightarrow G_{p+q}^{m+n}$ by

$$\alpha \otimes \beta \longrightarrow (\mu \wedge c)(\mathrm{id} \wedge t \wedge \mathrm{id})(\alpha \wedge \beta),$$

where t permutes the factors of $K_p \wedge TB^m$. Let G be the possibly nonassociative, bigraded ring $G = \Sigma G_p^m$. Denote multiplication by $\alpha \cdot \beta$.

We construct a function $\theta = \theta_V\colon H^*(M) \longrightarrow G$ as follows: Recall, there is a diagonal map $\Delta\colon T(\nu) \longrightarrow T(\nu) \wedge M^+$. On a vector $x \in \nu$, $x \longrightarrow (x, p(x))$ where p is the projection of ν. If $u \in H^p(M) = [M^+, K_p]$, $\theta(u) = ((V \wedge u)\Delta)_*(\alpha_M) \in G_p^m$.

By simply writing down the definitions one verifies

LEMMA (3.1). *If $u \otimes v \in H^*(M \times N)$ and U and V are B-Wu orientations of M and N, then*

$$\theta_{U \times V}(u \otimes v) = \theta_U(u) \cdot \theta_V(v).$$

The main work of this paper is in analyzing G. We postpone this to later sections and, at this point, give a theorem ((3.2)) which summarizes the results of this analysis.

Let Λ be the polynomial ring over $Z/4$ in indeterminants t and α_k, $k = 0, 1, 2, \ldots$, modulo the relations

$$t\alpha_k = \alpha_{k-1}, \quad \alpha_0 = 2, \quad 2t = 2\alpha_k = \alpha_k\alpha_l = 0.$$

Suppose $\lambda_p^{2p}\colon G_p^{2p} \longrightarrow Z/4$ and $\lambda_p^m\colon G_p^m \longrightarrow Z/2$, $m \neq 2p$, are homomorphisms. They give an additive map $\lambda\colon G \longrightarrow \Lambda$ as follows: If $\beta \in G_p^m$,

$$\begin{aligned} \lambda(\beta) &= \lambda_p^m(\beta) && \text{if } m = 2p, \\ &= \lambda_p^m(\beta)t^{2p-m} && \text{if } 2p > m, \\ &= \lambda_p^m(\beta)\alpha_{m-2p} && \text{if } 2p < m. \end{aligned}$$

Let $\varphi = \varphi_V\colon H^*(M) \longrightarrow \Lambda$ be defined by $\varphi = \lambda\theta$. If we let $\varphi_p^m = \lambda_p^m\theta$,

$$\varphi(u) = \varphi_{m/2}^m(u) + \sum_{2p>m} \varphi_p^m(u)t^{m-2p} + \sum_{2p<m} \varphi_p^m(u)\alpha_{m-2p},$$

$\varphi_{m/2}^m = 0$ if m is not even. Let $\mathrm{Sq} = \mathrm{Sq}^0 + \mathrm{Sq}^1 + \cdots$ be the Steenrod squaring operation. In §8 we prove

THEOREM (3.2). *Suppose $\{\zeta_m^k, \mu\}$ is a B-Wu system as in §2 and λ is as above. The μ and λ may be chosen so that for all B-Wu oriented manifolds M, φ_M satisfies:*

(i) $\varphi_p^m(u) = (\mathrm{Sq}\, u)(M)$ *if* $2p > m$,

$\varphi_p^m(u) = (\mathrm{Sq}\, u)(M) \mod (2)$ *if* $2p = m$.

(ii) $\varphi_p^m(u + v) = \varphi_p^m(u) + \varphi_p^m(v) + 2 \cdot (u \cup v)(M)$ *if* $2p \leq m$ *and* $u, v \in H^p(M)$. (*Note* $(u \cup v)(M) = 0$ *if* $2p \neq m$.)

(iii) *If ζ_k are oriented bundles, in the usual sense, and hence M has an orientation $\widetilde{M} \in H_m(M; Z)$ via its B-Wu orientation, and $m = 4l$, then $\varphi_{2l}^{4l}(u) = (\mathfrak{p}u)(\widetilde{M})$, where $\mathfrak{p}$ is the Pontrjagin square.*

(iv) φ_p^{2p} *is the function giving rise to* $\sigma(M)$ *in* [4].
(v) *If* $u \otimes v \in H^*(M \times N)$, $\varphi(u \otimes v) = \varphi(u)\varphi(v)$.

REMARK. The ring Λ arises from dividing G by the largest ideal possible without losing the quadratic property of φ. One may easily check that any proper ideal of Λ intersects $Z/4$ nontrivially and hence φ is linear modulo such an ideal.

REMARK. As we will see in §8, the choice of μ and λ are not independent. In fact, the choice of λ_p^{2p} for all p, determines the remaining choices.

We must next examine how φ depends on B-Wu orientation of M. Suppose U_1 and U_2: $\nu_M \longrightarrow \zeta_k^m$ are two B-Wu orientations of M which are equivalent as B structures, say to V: $\nu_M \longrightarrow \zeta_k$. Then U_1 is equivalent to (f_i, V) where f_i: $M \longrightarrow B_k^m$ are liftings of ν_M: $M \longrightarrow B_k$. f_1 and f_2 differ by a map x: $M \longrightarrow \Pi_{2i>m} K_{i-1}$. Conversely, U_1 and an x as above give a U_2 equivalent to U_1 as a B structure. In §9 we prove

THEOREM (3.3). *If* U_1 *and* U_2 *are B-Wu orientations of M which are equivalent as B structures on* ν_M *and* $x = \{x_i\}$, $x_i \in H^i(M)$, *is as above, then*

$$\varphi_{U_1}(u) - \varphi_{U_2}(u) = 2 \cdot (u \cup x_{[m/2]})(M) + \sum_{i>p} (u \cup x_i)(M)\alpha_{i-p}$$

where $u \in H^p(M)$.

DEFINITION (3.4). A B-Wu orientation U is *preferred* if $\varphi_U(H^p(M)) = 0$ for $2p < m$.

COROLLARY (3.5). *If* V: $\nu_M \longrightarrow \zeta_k$ *is a B structure on* ν_M, *V lifts to a preferred B-Wu orientation on M. The product of preferred orientations is preferred. Two preferred orientations coming from V differ by a trivialization of* v_s, $s = [m/2] + 1$.

PROOF. Suppose U_1 is any B-Wu orientation associated to V. By Poincaré duality, if $2p < m$, $\varphi_p(u) = (u \cup x_{m-p})(M)$ for some $x_{m-p} \in H^{m-p}(M)$. Change U_1 by $x = \{x_i\}$ to form U. By (3.4), U is preferred. The assertion about products follows from (3.2)(v). Note in the construction of U we have not used $x_{[m/2]}$. Uniqueness up to a trivialization of v_s now follows.

4. A generalization of the Kervaire invariant. We first recall some algebra from [4]. Suppose V is a vector space over $Z/2$ and φ: $V \longrightarrow Z_4$ is quadratic in the sense that $\varphi(u + v) - \varphi(u) - \varphi(v)$ is a nonsingular bilinear form with values in $Z/2 = \{0, 2\} \subset Z/4$. $\sigma(\varphi) \in Z/8$ is defined by the equation

$$\sum_{u \in V} i^{\varphi(u)} = \sqrt{2}^{\dim V}\left(\frac{i+1}{\sqrt{2}}\right)^{\sigma(\varphi)}$$

where $i = \sqrt{-1}$. In [4] it was proved:

THEOREM (4.0). *σ is additive on direct sums of quadratic functions and multiplicative on tensor products. If* $\varphi = 2\psi$, *where* ψ: $V \longrightarrow Z/2$, *is nonsingular quadratic, then* $\sigma(\varphi) = 4 \cdot \mathrm{Arf}(\psi)$. Arf θ *is* $\Sigma_i \psi(\lambda_i)\psi(\mu_i)$ *where* $\{\lambda_i, \mu_i\}$ *is a symplectic basis for V with respect to the bilinear form associated to* ψ. $\sigma(\varphi) = \dim V \bmod(2)$.

Suppose M and $\varphi: H^*(M) \longrightarrow \Lambda$ are as in (3.2). We utilize σ and φ to define invariants $\sigma(M) \in Z/8$ and $\sigma_k(M) \in Z/2$, $k = 1, 2, \ldots$. Let

$$\begin{aligned} V^k(M) &= H^n(M) && \text{if } m = 2n \text{ and } k = 0, \\ &= H^n(M) + H^{m-n}(M) && \text{if } m - k = 2n \text{ and } k > 0, \\ &= 0 && \text{otherwise.} \end{aligned}$$

Let $\theta^k\colon V^k(M) \longrightarrow Z/2$, for $k > 0$, be given by

$$\theta^k(u, v) = \varphi_k(u) + (\mathrm{Sq}\, v)(M) + (u \cup v)(M)$$

for $m - k$ even and $\theta^k = 0$ for $m - k$ odd.

Let $A = Z/8[a_k]$, $k = 1, 2, \ldots$, modulo the relations $a_k^2 = 4$, $2a_k = a_k a_l = 0$, $k \neq l$.

DEFINITION (4.1).

$$\begin{aligned} \sigma(M) &= \sigma(\varphi_n) && \text{if } m = 2n, \\ &= 0 && \text{if } m \text{ is odd.} \end{aligned}$$

$$\sigma_k(M) = \mathrm{Arf}(\theta^k), \quad \textstyle\sum(M) = \sigma(M) + \sum \sigma_k(M) a_k.$$

Our main result is

THEOREM (4.2). *If M and N are B-Wu oriented and $M \times N$ has the product orientation with respect to Wu system $\{\zeta_k^m, \mu\}$ where μ is as in* (3.2), *then* $\Sigma(M \times N) = \Sigma(M)\Sigma(N)$, *or equivalently*

$$\sigma(M \times N) = \sigma(M)\sigma(N) + 4\left(\sum \sigma_k(M)\sigma_k(N)\right),$$

$$\sigma_k(M \times N) = \sigma(M)\sigma_k(N) + \sigma_k(M)\sigma(N)$$

$(\sigma_k(M)\sigma(N) = \sigma_k(M)(\sigma(M) \bmod(2)))$, $\sigma(M) = \chi(M) \bmod(2)$ *where χ = Euler characteristic). If M and N have preferred orientations, $\sigma_k(M) = \sigma_k(N) = 0$, all k, and $\sigma(M \times N) = \sigma(M)\sigma(N)$.*

Before proving (4.2) we give some related results. σ is related to the Kervaire invariant and the index as follows:

THEOREM (4.3). *If the B-Wu orientation U of M has the property that $U_M\colon M \longrightarrow B_k^m$ is the constant map, that is, U is a framing and $m = 4l + 2$, then*

$$\sigma(M) = 4 \cdot (\textit{Kervaire invariant of } M).$$

If $\{\zeta_l\}$ are oriented in the usual sense and $m = 4l$, then

$$\sigma(M) = \mathrm{Index}(M) \bmod(8).$$

For any M, $\sigma(M) = \chi(M) \bmod(2)$, *where $\chi(M)$ is the Euler characteristic.*

The first and third parts of (4.3) are proved in [4] and the second part is a result of S. Morita [7].

THEOREM (4.4). *If $k > m$ or $m - k$ is odd, $\sigma_k(M) = 0$. If $m - k = 2n$, $\sigma_k(M)$*

$= \varphi_n(v_n)$ *where* v_n *is the nth Wu class. For any k,* $4 \cdot \sigma_k(M) = \sigma(\bar{S}^k \times M)$ *where* $\bar{S}^k$ *is* S^k *with the nontrivial B-Wu orientation as in* §2.

PROOF. The first part follows from the fact that $V^k(M) = 0$ for $k > m$ or $m - k$ odd.

Suppose $m - k = 2n$ and $u \in H^{m-n}(M)$. $(\mathrm{Sq}\, u)(M) = (u \cup v_n)(M)$. Choose a basis λ_i for $H^n(M)$ and μ_i for $H^{m-n}(M)$ such that $\lambda_i \cup \mu_j = \delta_{ij}$. $\{\lambda_i, \mu_i\}$ is a symplectic basis for $V^k(M)$. If $v_n = 0$, $\theta^k(\mu_i) = (\mathrm{Sq}\, \mu_i)(M) = 0$, and hence $\mathrm{Arf}\, \theta^k = 0$. If $v_n \neq 0$, choose $\lambda_1 = v_n$. Then

$$\mathrm{Arf}\, \theta^k = \sum \theta^k(\lambda_i)\theta^k(\mu_i) = \theta^k(v_n) = \varphi_n(v_n).$$

In §9 we will prove

LEMMA (4.5). *For* $1 \in H^0(\bar{S}^k)$, $\varphi(1) \neq 0$. *Hence* $\sigma_k(\bar{S}^k) = \varphi_k(v_0) = \varphi_k(1) = 1$. *Then by* (4.3), $\sigma(\bar{S}^k \times M) = 4 \cdot \sigma_k(M)$.

Combining the above results yields:

COROLLARY (4.6). $\sigma_k(\bar{S}^k) = 1$, $\sigma(\bar{S}^k \times \bar{S}^k) = 4$, $\sigma_k(\bar{S}^k \times P_{2n}) = 1$, *where* P_{2n} *is real projective 2n-space.*

The last equation in (4.6) is proved as follows:

$$\sigma_k(\bar{S}^k \times P_{2n}) = \sigma_k(\bar{S}^k)\sigma(P_{2n}) = \sigma_k(\bar{S}^k)\chi(P_{2n}) = 1.$$

In our language, W. Browder's formula [2] is:

COROLLARY (4.7). *If the Wu classes* v_i, $i > 0$, *are zero on M and* $\varphi_M(1) = 0$, *then* $\sigma(M \times N) = \sigma(M)\chi(N)$.

PROOF. By (4.4) $\sigma_k(M) = 0$ for all k. (4.7) is obvious if M is odd-dimensional. Suppose $m = 2l$. Then by (3.2)(ii), on M,

$$2\varphi_l(u) = 2 \cdot (u \cup u)(M) = 2 \cdot (v_n \cup u)(M) = 0.$$

Hence $\varphi_l(u) \in \{0, 2\}$ and $\varphi_l = 2\theta$, where $\theta: H^l(M) \longrightarrow Z/2$. Thus $\sigma(M) = 4 \cdot \mathrm{Arf}\, \theta \in \{0, 4\}$. Then

$$\sigma(M \times N) = \sigma(M)\sigma(N) = \sigma(M)(\sigma(N) \bmod (2)) = \sigma(M)\chi(N).$$

Finally, we sketch a proof of Sullivan's product formula for surgery obstructions. Suppose $X = (x, \eta, \alpha)$ is a Poincaré complex as in §1, X is simply connected and has formal dimension $4l + 2$, M is a manifold and $f: \nu_M \longrightarrow \eta$ is a map such that $T(f)_*\alpha_M = \alpha$. Choose a BG-Wu orientation of X and orient M by pulling back this orientation via f. In [4] it is shown that the surgery obstruction to making $f_M: M \longrightarrow X$ into a homotopy equivalence, $\sigma(f) \in Z/2$, is given by

$$4 \cdot \sigma(f) = \sigma(M) - \sigma(X) \in \{0, 4\} \subset Z/8.$$

Suppose N, Y and g satisfy the same hypotheses as M, X and f except that $\dim N = \dim Y = 4k$.

COROLLARY (4.8). $\sigma(f \times g) = \sigma(f)\chi(Y)$.

PROOF. Recall,

$$\sigma(Y) - \sigma(N) = I(Y) - I(N) \mod(8),$$
$$= 0 \qquad \mod(8),$$

since $I(Y) - I(N)$, the surgery obstruction of g, is divisible by 8. As in (4.4), $\sigma_k(X) = \varphi_n(v_n(X))$. $\varphi_X = \varphi_M f_M^*$ (see [4]). Thus $\sigma_k(M) = \sigma_k(X)$ and $\sigma_k(N) = \sigma_k(Y)$. Hence

$$4 \cdot \sigma(f \times g) = \sigma(M \times N) - \sigma(X \times Y) = \sigma(M)\sigma(N) - \sigma(X)\sigma(Y)$$
$$= (\sigma(M) - \sigma(X))\sigma(Y) + \sigma(M)(\sigma(N) - \sigma(Y))$$
$$= (4 \cdot \sigma(f))\sigma(Y) = 4 \cdot (\sigma(f)\chi(Y)),$$

since $\sigma(Y) = \chi(Y) \mod(2)$. Hence $\sigma(f \times g) = \sigma(f)\chi(Y)$.

PROOF OF (4.2). We first prove

(4.9) $$\sigma_k(M \times N) = \sigma(M)\sigma_k(N) + \sigma_k(M)\sigma(N).$$

Since $\sigma(M) = \chi(M) \mod(2)$, (4.9) is equivalent to

(4.10) $$\sigma_k(M \times N) = \chi(M)\sigma_k(N) + \sigma_k(M)\chi(N).$$

If $m + n - k$ is odd, one easily checks that both sides of (4.10) are zero. Suppose $2p = m + n - k$. Note that the proof in (4.4) that $\sigma_k(M) = \varphi_l(v_l)$ for $2l = m - k$ does not depend on (4.2).

$$v_p(M \times N) = \sum v_i(M) \otimes v_{p-i}(N)$$

and

$$\sigma_k(M \times N) = \varphi_p(v_p(M \times N)).$$

Since $v_i(M) = 0$, $2i > m$, the above sum for $v_p(M \times N)$ has three types of nonzero terms: (a) $i = m/2$; (b) $i < m/2, p - i = n/2$; (c) $i < m/2, p - i < n/2$. In case (c),

$$\varphi(v_i \otimes v_{p-i}) = \varphi_i(v_i)\alpha_{m-2i}\varphi_{p-i}(v_{p-i})\alpha_{n-2(p-i)} = 0,$$

since $\alpha_k\alpha_l = 0$ in Λ. In case (a),

$$\varphi(v_i \otimes v_{p-i}) = \varphi_{m/2}(v_{m/2})\varphi_{p-i}(v_{p-i})\alpha_{n-2(p-i)}.$$

When $i = m/2$, $p - i = (n - k)/2$ and hence $\varphi_{p-i}(v_{p-i}) = \sigma^k(N)$. Let $l = m/2$. By (3.2)(i),

$$\varphi_l(v_l) = (v_l^2)(M) \mod(2).$$

But $(v_l^2)(M) = \chi(M) \mod(2)$. The same argument applies to case (b), and (a) and (b) give (4.10).

We complete the proof of (4.2) by showing

(4.11) $$\sigma(M \times N) = \sigma(M)\sigma(N) + 4 \cdot \left(\sum \sigma_k(M)\sigma_k(N)\right).$$

Let $V_1^k = V^k(M)$, $V_2^k = V^k(N)$ and $V^k = V^k(M \times N)$. Let $\varphi_1^0 = \varphi_M | V^0(M)$ and $\varphi_1^k = 2\theta_M^k$ for $k > 0$. Let φ_2^k and φ^k be similarly defined for N and $M \times N$ and let $\bar{\varphi} = \Sigma \varphi^k$. Since σ is additive,

(4.12) $$\sigma(\bar{\varphi}) = \sum \sigma(\varphi^k) = \sigma(M \times N) + 4 \cdot \sum \sigma^k(M \times N).$$

On the other hand, $H^*(M \times N) = \Sigma V_1^k \otimes V_2^l$ and hence

$$\sigma(\bar{\varphi}) = \sum \sigma(\bar{\varphi}|V_1^k \otimes V_2^l); \tag{4.13}$$

$\bar{\varphi}|V_1^0 \otimes V_2^0 = \varphi_1^0\varphi_2^0$ and therefore

$$\sigma(\bar{\varphi}|V_1^0 \otimes V_2^0) = \sigma(\varphi_1^0\varphi_2^0) = \sigma(\varphi_1^0)\sigma(\varphi_2^0) = \sigma(M)\sigma(N). \tag{4.14}$$

We show that for k and l not both zero,

$$\begin{aligned} \sigma(\bar{\varphi}|V_1^k \otimes V_2^l) &= 4 \cdot (\sigma_k(M)\sigma_k(N)) && \text{if } k = l, \\ &= 4 \cdot (\chi(M)\sigma_l(N)) && \text{if } k = 0, \\ &= 4 \cdot (\sigma_k(M)\chi(N)) && \text{if } l = 0, \\ &= 0 && \text{otherwise.} \end{aligned} \tag{4.15}$$

Combining this with (4.12), (4.13), and (4.14) we obtain (4.11). To prove (4.15) one chooses symplectic bases for V_1^k and V_2^l as in the proof of 4.4 for k and $l > 0$. One then forms a symplectic basis for $V_1^k \otimes V_2^l$ and computes the Arf invariant of $(\bar{\varphi}|V_1^k \otimes V_2^l)/2$. We leave the details to the reader.

5. **The ring** G. Let TB^m and μ be as in §2. Let $G_{n,p}^m = \pi_n(TB^m \wedge K_p)$, $G = \Sigma G_p^m$, where $G_p^m = G_{m,p}^m$. As in §3, μ and the cup product maps define a pairing $\bar{\mu}$: $G_{n,p}^m \otimes G_{s,q}^n \longrightarrow G_{n+s,p+q}^{m+n}$. Let $\alpha \cdot \beta = \bar{\mu}(\alpha \otimes \beta)$. The main objective of this section is to compute $G_{n,p}^m$ for $2p \geqslant n$ and to describe $\bar{\mu}$. To do this we introduce a variety of gadgets.

For any space or spectrum X let ρ: $\pi_m(X \wedge K_p) \longrightarrow H_{m-p}(X)$ be defined by $u(\rho(\alpha)) = (u \otimes \iota_p)(\alpha)$ for all $u \in H^{m-p}(X)$ (ρ is the Hurewicz map followed by $/\iota_p$). ι_p is the generator of $H^p(K_p)$. ρ is the map which gives an isomorphism $\lim_{p\to\infty}\pi_{n+p}(X \wedge K_p) \approx H_n(X)$. Also let $\tau(l)$: $\pi_m(X \wedge K_p) \longrightarrow \pi_{m+l}(X \wedge K_{p+l})$ be the maps used to define the above limit, that is $\tau(l)$ is the composition of the obvious maps

$$\pi_m(X \wedge K_p) \approx \pi_{m+l}(X \wedge S^l K_p) \longrightarrow \pi_{m+l}(X \wedge K_{p+l}).$$

Let s: $G_{n,p}^m \longrightarrow Z/2$ be given by $s(\alpha) = (1 \otimes \mathrm{Sq}\, \iota_p)(\alpha)$ where $1 \in H^0(TB^m)$ is the Thom class. Let S be the sphere spectrum. Recall $\pi_{2p}(S \wedge K_p) \approx Z/2$. Let $\beta_p^m \in G_{2p,p}^m$ be the image of the generator under

$$(i \wedge \mathrm{id})_*: \pi_{2p}(S \wedge K_p) \longrightarrow \pi_{2p}(TB^m \wedge K_p)$$

where i: $S \longrightarrow TB^m$ is the "inclusion of a fibre" and $p > 0$. Let $\beta_0^m = 2[i] \in \pi_0(TB^m) = G_{0,0}^m$.

LEMMA (5.1).

(i)

$$\tau(l)(\alpha) \cdot \tau(k)(\beta) = \tau(l+k)(\alpha \cdot \beta), \quad s(\alpha \cdot \beta) = s(\alpha)s(\beta),$$

$$s(\alpha) = 0 \quad \textit{if } \alpha \in G_{n,p}^m \textit{ and } n > 2p,$$

$$s\tau(\alpha) = s(\alpha), \quad s(\alpha) = (\chi(\mathrm{Sq})1)(\rho(\alpha)), \quad \rho\tau(l) = \rho.$$

(ii) *If* $2p > n$, ρ: $G_{n,p}^m \longrightarrow H_{n-p}(TB^m)$ *is an isomorphism. If* $2p = n$, ρ *is an* *epimorphism and the kernel is* $\{0, \beta_p^m\}$. $\beta_p^m \neq 0$ *if* $m < 2(p+1)$.

(iii) *If* $\beta \in G^n_{2q,q}$, $\beta_p^m \cdot \beta = \beta \cdot \beta_p^m = s(\beta)\beta^{n+m}_{p+q}$. *Note* $G^0_{00} = \pi_0(S) = Z$ *and* $n \cdot \beta = n\beta$, $n \in Z$. *Hence a special case of* (iii) *is* $2\beta = \beta_0^0 \cdot \beta = s(\beta)\beta_q^n$.

PROOF. (i) follows directly from the definitions. For large l, ρ: $\pi_{n+l}(TB^m \wedge K_{p+l}) \to H_{n-p}(TB^m)$ is an isomorphism since TB^m is connected. TB^m is (-1) connected and $K_{p+l} = S^lK_p \cup e^{2p+1+l} \cup \cdots$. Hence, $\tau(l)$: $\pi_n(TB^m \wedge K_p) \approx \pi_{n+l}(TB^m \wedge K_{p+l})$ for $2p > n$. Therefore ρ is an isomorphism on $G^m_{n,p}$ for $2p > n$.

Let $i = S\iota_p$: $SK_p \to K_{p+1}$ be an inclusion, $D = K_{p+1}/SK_p$, and j: $K_{p+1} \to D$ the quotient map. D is $2p+1$ connected, kernel of $H^{2p+2}(i)$ is $\mathrm{Sq}^{p+1}\iota_{p+1}$ and $H^{2p+2}(D) \approx Z/2$ is generated by a class γ_p such that $H^{2p+2}(j)\gamma = \mathrm{Sq}^{p+1}\iota_{p+1}$. Consider the commutative diagram: Denote TB^m by T^m.

$$\begin{array}{ccccccccc}
\pi_{2p+2}(T^m \wedge K_{p+1}) & \longrightarrow & \pi_{2p+2}(T^m \wedge D) & \xrightarrow{\partial^*} & \pi_{2p+1}(T^m \wedge SK_p) & \longrightarrow & \pi_{2p+1}(T^m \wedge K_{p+1}) & \longrightarrow & 0 \\
\approx\downarrow\rho & & \downarrow\bar{\gamma} & & \approx\uparrow S & & \approx\downarrow\rho & & \\
H_{p+1}(T^m) & \xrightarrow{\chi(\mathrm{Sq}^{p+1})} & H_0(T^m) & & \pi_{2p}(T^m \wedge K_p) & \xrightarrow{\rho} & H_p(T^m) & &
\end{array}$$

where $u(\bar{\gamma}(\alpha)) = (u \otimes \gamma_p)(\alpha)$ for all $u \in H^0(T^m)$. The top row is exact and the squares commute. Since D is $2p+1$ connected and $\pi_{2p+2}(D) \approx Z/2$, for $p > 0$, the Hurewicz theorem gives that $\bar{\gamma}$ is an isomorphism for $p > 0$. Therefore the kernel of ρ: $\pi_{2p}(T^m \wedge K_p) \to H_p(T^m)$ is image$(S^{-1})\partial_*$. The above diagram is functorial in T^m so i: $S \to T^m$ carries the diagram for S into the diagram for T^m. For the S diagram, image $S^{-1}\partial_* = \pi_{2p}(S \wedge K_p)$. Hence for T^m, ker ρ is $\{0, \beta_p^m\}$, $p > 0$. In the above diagram

$$\chi(\mathrm{Sq}^{p+1})(c) = (\chi(\mathrm{Sq}^{p+1})1)(c) = v_{p+1}(c).$$

Therefore (ii) is proved for $p > 0$. For $p = 0$, $\pi_{2p+2}(T^m \wedge D) \approx \pi_{2p}(T^m)$ and $S^{-1}\partial_*$ is multiplication by 2. Thus ker ρ is $2[i] = \beta_0^m$ and (ii) is proved for $p = 0$.

To prove (iii) consider the commutative diagram: $(D_p = K_{p+1}/SK_p)$.

$$\begin{array}{ccccc}
K_q \wedge SK_p & \xrightarrow{\mathrm{id} \times i} & K_q \wedge K_{p+1} & \xrightarrow{\mathrm{id} \times j} & K_q \wedge D_p \\
\downarrow a & & \downarrow b & & \downarrow c \\
SK_{p+q} & \xrightarrow{i} & K_{p+q+1} & \xrightarrow{j} & D_{p+q}
\end{array}$$

where a is the suspension of the cup product map, b is cup product and c is defined by b. (Choose (b, a) to be an inclusion of pairs.) $H^*(jb)(\gamma_{p+q}) = \mathrm{Sq}^q\iota_q \otimes \mathrm{Sq}^{p+1}\iota_{p+1}$. Hence,

$$H^*(c)\gamma_{p+q} = \mathrm{Sq}^q\iota_q \otimes \gamma_p. \tag{5.2}$$

Let $\zeta_p^m \in \pi_{2p+2}(T^m \wedge D_p)$ be the generator. $\partial_*\zeta_p^m = \beta_p^m$. μ and c define a map $T^n \wedge K_q \wedge T^m \wedge D_p \to T^{n+m} \wedge D_{p+q}$. By (5.2), this map takes $\beta \wedge \zeta_p^m$ into $s(\beta)\zeta^{n+m}_{p+q}$ for $\beta \in G^n_{2q,q}$. Smash the above diagram with

$$\begin{array}{c} T^n \wedge T^m \\ \downarrow \\ T^{n+m} \end{array}$$

and apply π_* to obtain a commutative ladder of exact sequences with connecting homomorphisms ∂_*

$$\bar{\mu}(\beta \otimes \beta_p^m) = \bar{\mu}(\partial_*(\beta \otimes \zeta_p^m)) = \partial_*(s(\beta)\zeta_{p+q}^{m+n}) = s(\beta)\beta_{p+q}^{m+n}.$$

This completes the proof of (5.1).

6. **The ring** L. The ring G is unnecessarily complicated for our purposes. For example G_p^m for low values of p is very messy and θ: $H^p(M) \longrightarrow G_p^m$ is neither linear nor quadratic. In this section we introduce a simpler ring L. Let

$$\begin{aligned} L_p^m &= G_{m,p}^m = G_p^m && \text{if } 2p > m, \\ &= G_{2(m-p),m-p}^m && \text{if } 2p \leqslant m, \\ L &= \sum L_p^m. \end{aligned}$$

Let $\alpha_p^m \in L_p^m$ be defined by $\alpha_p^m = \beta_{m-p}^m$ if $2p \leqslant m$ and $\alpha_p^m = 0$ if $2p > m$. Let g: $G \longrightarrow L$ by

$$\begin{aligned} g(\beta) &= \beta && \text{if } \beta \in G_p^m \text{ and } 2p > m, \\ &= \tau(m-2p)(\beta) && \text{if } 2p \leqslant m. \end{aligned}$$

We define a multiplication of L so that g is a ring homomorphism as follows: Suppose $\alpha \in L_p^m$ and $\beta \in L_q^n$. If $2p > m$ and $2q > n$ or $2p \leqslant m$ and $2q \leqslant n$, $\alpha\beta = \alpha \cdot \beta$. Otherwise $\alpha\beta$ is defined as follows: Let

$$\begin{aligned} r(\alpha) &= \alpha && \text{if } 2p \leqslant m, \\ &= \tau(2p-m)^{-1}\alpha && \text{if } 2p > m. \end{aligned}$$

If $\gamma \in G_{a,b}^m$, let $t(m, p)\gamma = \tau(l)\gamma$ where l is chosen, if possible, so that $\tau(l)\gamma \in L_p^m$; then $\alpha\beta = t(m+n, p+q)(r(\alpha) \cdot r(\beta))$. It is extremely tedious but straightforward, using (5.1)(ii) and (iii), to check that $\alpha\beta$ is well defined. The formula $\tau(l)(\alpha) \cdot \tau(k)(\beta) = \tau(l+k)(\alpha \cdot \beta)$ immediately yields:

LEMMA (6.1). $g(\alpha \cdot \beta) = \alpha\beta$.

LEMMA (6.2). $0 \longrightarrow \{\alpha_p^m\} \longrightarrow L_p^m \xrightarrow{p} H_{m-p}(T^m) \longrightarrow 0$ *is exact.* $\alpha_p^m \neq 0$ *if* $2p \leqslant m$. *If* $\beta \in L_p^m$,

$$\alpha_q^n\beta = \beta\alpha_q^n = s(\beta)\alpha_{p+q}^{n+m}, \quad 2\beta = s(\beta)\alpha_p^m.$$

PROOF. (5.1)(ii) gives the exactness above. Suppose $2q > n$ and hence $\alpha_q^n = 0$. If $2p < m$, $s(\beta) = 0$. If $2p \geqslant m$, $\alpha_{p+q}^{n+m} = 0$. Suppose $2q \leqslant n$. If $2p \leqslant m$,

$$\alpha_q^n\beta = \beta_{n-q}^n \cdot \beta = s(\beta)\beta_{n-q+m-p}^{n+m} = s(\beta)\alpha_{p+q}^{n+m}.$$

If $2p > m$,

$$\alpha_q^n\beta = t(m+n, p+q)(\alpha_q^n \cdot r(\beta)).$$

If $2(p+q) > m+n$ this is zero and also $\alpha_{p+q}^{n+m} = 0$. If $2(p+q) \leqslant m+n$, $t(m+n, p+q) = \mathrm{id}$ and

$$\alpha_q^n \cdot r(\beta) = s(r(\beta))\alpha_{p+q}^{n+m} = s(\beta)\alpha_{p+q}^{n+m}.$$

If $2p > m$, $2\beta = 0$ since $L_p^m \approx H_{m-p}(T^m)$. If $2p \leqslant m$, $2\beta = s(\beta)\alpha_p^m$ by (5.1)(iii).

7. **Quadratic functions with values in** L. Let M be a B-Wu oriented manifold and let $\theta\colon H^*(M) \longrightarrow G$ be as in §3. Define $\psi\colon H^*(M) \longrightarrow L$ by $\psi = g\theta$ where $g\colon G \longrightarrow L$ is the map given in (6.1).

PROPOSITION (7.1). *ψ has the following properties*:

(i) $\psi(u+v) = \psi(u) + \psi(v) + (u \cup v)(M)\alpha$ *where* $u, v \in H^p(M)$, $\alpha = 0$ *if* $m \neq 2p$ *and* $\alpha = \alpha_p^{2p}$ *if* $m = 2p$.

(ii) *If* $u \otimes v \in H^*(M \times N)$, $\psi(u \otimes v) = \psi(u)\psi(v)$.

(iii) *If U is the B-Wu orientations of M, then the diagram below is commutative.*

$$\begin{array}{ccc} H^p(M) & \xrightarrow{\ \psi\ } & L_p^m \\ \big\downarrow{\scriptstyle \cap M} & & \big\downarrow{\scriptstyle t} \\ H_{m-p}(M) & \xrightarrow{(U_M)_*} & H_{m-p}(B^m) \end{array}$$

$t = \rho d$ and $d\colon H_*(TB^m) \approx H_*(B^m)$ is the Thom isomorphism.

PROOF. The definitions of t and ψ immediately yield (iii). (ii) follows from (3.1) and (6.2). (i) is true for $2p > m$ by (iii) and the fact that t is an isomorphism for $2p > m$.

To prove (i) for $2p \leqslant m$ consider

$$A\colon H^p(M) \approx H^{m-p}(S^{m-2p}M^+) = [S^{m-2p}M^+, K_{m-p}] \longrightarrow \{S^{m-2p}M^+, K_{m-p}\}$$

where $\{\ ,\ \}$ denotes stable maps. In [4] it is shown that A is quadratic and its associated bilinear form is cup product in $H^*(S^{m-2p}M^+)$. Thus A is linear if $m \neq 2p$.

$$S \xrightarrow{\alpha_M} T(\nu_M) \xrightarrow{\Delta} T(\nu_M) \wedge M^+$$

is an S-duality giving an S duality isomorphism

$$B\colon \{S^{m-2p}M^+, K_{m-p}\} \approx \pi_M(T(\nu_M) \wedge K_{m-p}).$$

The B-Wu orientation U on M gives a map

$$(U \wedge \mathrm{id})_*\colon \pi_m(T(\nu_M) \wedge K_{m-p}) \longrightarrow \pi_m(TB^m \wedge K_{m-p}) = L_p^m,$$

$\psi = (U \wedge \mathrm{id})_* BA$. Hence ψ is linear for $2p < m$. In [4] it is shown that ψ satisfies (i) when $m = 2p$.

8. **Proof of (3.2).** Recall, in §2, B^m was constructed by killing v_i for $2i > m$. We may arrange these constructions so that $pt = B^0 \subset B^1 \subset \cdots \subset B^m \subset \cdots$. These maps define inclusion maps $j_l\colon TB^m \longrightarrow TB^{m+l}$. For $2p \leqslant m$, j_{m-2p} induces a

map

$$r\colon L_p^m = \pi_{2(m-p)}(TB^m \wedge K_{m-p}) \longrightarrow \pi_{2(m-p)}(TB^{2(m-p)} \wedge K_{m-p}) = L_{m-p}^{2(m-p)}.$$

Furthermore, since α_p^m comes from $S \wedge K_{m-p} \longrightarrow TB^m \wedge K_{m-p}$, $r(\alpha_p^m) = \alpha_{m-p}^{2(m-p)}$.

LEMMA (8.1). *If the Wu system* $\{\zeta_k^m, \mu\}$ *is oriented in the usual sense,* $(X \otimes \mathfrak{p}\, \iota_{2l})(\alpha_{2l}^{4l}) = 2$ *where* $\mathfrak{p}$ *is the Pontrjagin square and* $X \in H^0(TB^m; Z)$ *is the Thom class.*

PROOF. Put the trivial B-Wu orientation on $M = S^{2l} \times S^{2l}$ and let $u = x \otimes 1 + 1 \otimes x$, where $x \in H^{2l}(S^{2l})$ is the generator. Since u comes from an integer class, $\mathfrak{p}u = u^2$ and $\mathfrak{p}u(M) = 2(x \otimes x)(M) = 2$. $M \longrightarrow B^m$ is trivial and therefore, by (7.1)(iii), $\psi(u) = c\alpha_l^{4l}$, $c \in Z/2$. $(X \otimes \mathfrak{p}\iota_{2l})(\psi(u)) = \mathfrak{p}(u)(M) = 2$. Thus $c = 1$ and (8.1) is proved.

Choose homomorphisms $h_n\colon L_n^{2n} \longrightarrow Z/4$ such that $h_n(\alpha_n^{2n}) = 2$. This is possible by (6.2). If $\{\zeta_k^m\}$ are oriented, choose $h_{2l} = (X \otimes \mathfrak{p}\iota_{2l})$. Recall, by (6.2), if $\beta \in L_p^m$, $2\beta = s(\beta)\alpha_p^m$ and $s(\beta) = 0$ for $2p < m$. Thus every element of L_p^m, for $2p < m$, has order two. We define λ_p^m as in §3 by the formulas:

$$\begin{aligned} \lambda_p^m &= s && \text{if } 2p > m,\\ &= h_p && \text{if } 2p = m,\\ 2 \cdot \lambda_p^m &= h_{m-p}rg && \text{if } 2p < m, \end{aligned}$$

where r is as above and $g\colon G \longrightarrow L$ as in §6. Equivalently stated,

$$\begin{aligned} \varphi_p(u) &= s\psi(u) && \text{if } 2p > m,\\ &= h_p\psi(u) && \text{if } 2p = m,\\ 2 \cdot \varphi_p(u) &= h_{m-p}r\psi(u) && \text{if } 2p < m. \end{aligned}$$

LEMMA (8.2). φ_p *satisfies* (3.2)(i)–(iv).

PROOF. Proof of (i): If $2p > m$,

$$\begin{aligned} \varphi_p(u) &= s\psi(u) = (1 \otimes \mathrm{Sq}\, \iota_p)(\psi(u))\\ &= (1 \otimes \mathrm{Sq}\, \iota_p)((U \wedge u)\Delta)_*(\alpha_M)\\ &= ((\mathrm{Sq}\, u) \cup X_2)(\alpha_M) = (\mathrm{Sq}\, u)(M). \end{aligned}$$

X_2 = Thom class of $T(\nu_M)$. To show that $\varphi_p(u) = (\mathrm{Sq}\, u)(M) \mod(2)$ for $m = 2p$, we show that $h_p = s \mod(2)$ and hence the above argument gives the desired result.

$$2h_p(\beta) = h_p(2\beta) = h_p(s(\beta)\alpha_p^{2p}).$$

Since $h_p(\alpha_p^{2p}) = 2$, $h_p = s \mod(2)$.

(3.2)(ii) follows from (7.1)(i). (iii) is true by construction. (iv) holds because our construction of φ_p for $2p = m$ is exactly the same as in [4].

To complete the proof of (3.2) we show that μ can be chosen so that $\varphi(u \otimes v) = \varphi(u)\varphi(v)$. $\varphi = \lambda\theta$ and $\theta(u \otimes v) = \theta(u) \cdot \theta(v)$ by (3.1). Also $\lambda = \bar{\lambda}g$ where $\bar{\lambda}\colon L \longrightarrow$

Λ. g is multiplicative. Denote $\bar{\lambda}$ by λ. (3.2)(v) will be proved if we prove

LEMMA (8.3). *The maps μ for the B-Wu system $\{\zeta_k^m, \mu\}$ may be chosen so that $\lambda\colon L \longrightarrow \Lambda$ satisfies $\lambda(\alpha\beta) = \lambda(\alpha)\lambda(\beta)$.*

In the remainder of this section $\alpha \in L_p^m$ and $\beta \in L_q^n$, $r = p + q$ and $l = n + m$. We first prove that (8.3) holds for $2r > l$ and for $\alpha = \alpha_p^m$ or $\beta = \alpha_q^n$ independently of the choice of μ. If $2r > l$,

$$\lambda(\alpha\beta) = s(\alpha\beta)t^{2r-l} = s(\alpha)s(\beta)t^{2r-l}.$$

If $2p < m$, $s(\alpha) = 0$ and

$$\lambda(\alpha)\lambda(\beta) = (\lambda_p^m(\alpha)\alpha_{m-2p})(s(\beta)t^{2q-n}) = 0,$$

since $m - 2p < 2q - n$ and thus $\alpha_{m-2p}t^{2q-n} = 0$. If $2p = m$,

$$\lambda(\alpha)\lambda(\beta) = \lambda_p^m(\alpha)(s(\beta)t^{2q-n}) = (\lambda_p^m(\alpha) \bmod(2))s(\beta)t^{2q-n}.$$

As shown in the proof of (8.2), $\lambda_p^m = h_p = s \mod(2)$. Hence $\lambda(\alpha)\lambda(\beta) = s(\alpha)s(\beta)t^{2q-n}$. The same argument holds for $2q \leqslant n$. If both $2p > m$ and $2q > n$, $\lambda(\alpha)\lambda(\beta) = s(\alpha)s(\beta)t^{2r-l}$.

Suppose $\alpha = \alpha_p^m \cdot \alpha_p^m \beta = s(\beta)\alpha_r^l$. If $2p < m$ or $2l > r$, $s(\beta)\alpha_r^l = 0$ and $\lambda(\alpha_p^m)\lambda(\beta) = \alpha_{m-2p}s(\beta)t^{2q-n} = 0$. If $2p = m$ and $2r < l$, $s(\beta) = 0$ and $\lambda(\alpha_r^l)\lambda(\beta) = 2\lambda_q^n(\beta)\alpha_{n-2q} = 0$. If $2p = m$ and $2r = l$, $\lambda(s(\beta)\alpha_r^l) = 2 \cdot s(\beta)$. $\lambda(\alpha_r^l)\lambda(\beta) = 2 \cdot s(\beta)$ because $\lambda(\alpha_r^l) = 2$ and $\lambda(\beta) = s(\beta) \mod(2)$.

$\{\alpha_r^l\}$ is the kernel of $\rho\colon L_p^m \longrightarrow H_{m-p}(TB^m)$. Hence we have proved

LEMMA (8.4). *There is a class $d \in \Sigma H^i(TB^m \wedge TB^n)$ where the sum is over n and m and $i \geqslant [n + m/2]$, such that*

$$\lambda(\alpha\beta) - \lambda(\alpha)\lambda(\beta) = d(\rho(\alpha) \otimes \rho(\beta))\alpha_{l-2r}.$$

(If $l = 2r$, $\alpha_0 = 2$ means $2 \cdot$.)

Let μ' be some choice of μ. With respect to μ' we obtain a d as in (8.4). Let $\bar{d} \in \Sigma H^*(B^m \times B^n)$ be the class corresponding to d under the Thom isomorphism. B^{m+n} is a principal fibration over B with fibre ΠK_i, $2(i + 1) > m + n$. Modify $B^m \times B^n \longrightarrow B^{m+n}$ by d to obtain a new $\mu\colon TB^m \wedge TB^n \longrightarrow TB^{m+n}$. Consider the bilinear form from $L_p^m \otimes L_q^n$ to Λ defined by $\bar{A}(\alpha, \beta) = \lambda(\alpha\beta) - \lambda(\alpha * \beta)$ where $\alpha\beta$ comes from μ and $\alpha * \beta$ from μ'. As proved above, $\lambda(\alpha\beta) = \lambda(\alpha)\lambda(\beta)$ for $2r > l$ and for $\alpha = \alpha_p^m$ or $\beta = \alpha_q^n$, independently of the choice of μ. Hence $\bar{A}(\alpha, \beta) = 0$ if $2r > l$ and $\bar{A}(\alpha, \beta) = A(\rho(\alpha), \rho(\beta))$ where A is a bilinear form from $H_{m-p}(TB^m) \otimes H_{n-q}(TB^n)$ to Λ. Suppose $2r \leqslant l$. Consider the commutative diagram:

$$\begin{array}{ccccc}
L_p^m \otimes L_q^n & \longrightarrow & L_r^l & \longrightarrow & L_{l-r}^{2(l-r)} \\
\downarrow{\scriptstyle \rho \otimes \rho} & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho} \\
H_{m-p}(TB^m) \otimes H_{n-q}(TB^n) & \longrightarrow & H_{l-r}(TB^l) & \longrightarrow & H_{l-r}(TB^{2(l-r)})
\end{array}$$

The bottom row is independent of μ since $H_{l-r}(TB^{2(l-r)}) \approx H_{l-r}(B)$. Therefore

$$A(\rho(\alpha), \rho(\beta)) = D(\rho(\alpha), \rho(\beta))\alpha_{l-2r}$$

where D takes values in $Z/2$. To complete the proof of (8.2) we show

LEMMA (8.5). $D(\rho(\alpha), \rho(\beta)) = d(\rho(\alpha) \otimes \rho(\beta))$.

PROOF. Let $\mathbf{Y}^l$ be the spectrum described in (2.3), that is, Y^l_k is the two stage Postnikov system with k-invariant $\Pi_{2i>l}\chi(\mathrm{Sq}^i)$: $K_k \longrightarrow \Pi K_{k+i}$. Let $\mathbf{K}_i$ be the spectrum whose jth term is K_{i+j} and $\iota_i \in H^i(\mathbf{K}_i)$ be the generator. Also denote the generator of $H^0(\mathbf{Y}^l)$ by ι_0. Let f^l: $TB^l \longrightarrow \mathbf{Y}^l$ be a map such that $f * \iota_0$ is the Thom class. f exists because $\chi(\mathrm{Sq}^i)1 = v_i \cup 1$. One may check that for any space X, if G': $X \longrightarrow B^l$ and one modifies G' by $\bar{d} = \{d_i\}$, $d_i \in H^i(X)$, $2i > l$ to form G: $X \longrightarrow B^l$, then $T(G * \zeta^l) \longrightarrow TB^l \longrightarrow \mathbf{Y}^l$ is modified by $d_i \in H^i(TG * \zeta^l)$ where d_i corresponds to $\bar{d}_i$ under the Thom isomorphism. We may choose f^l and maps g_2: $\mathbf{Y}^l \longrightarrow \mathbf{Y}^{l+t}$ such that

$$\begin{array}{ccc} \prod_{2(i+1)>l} \mathbf{K}_i & \xrightarrow{g_1} & \prod_{2(i+1)>l-r} \mathbf{K}_i \\ \downarrow & & \downarrow \\ \mathbf{Y}^l & \xrightarrow{g_2} & \mathbf{Y}^{2(l-r)} \\ \uparrow & & \uparrow \\ TB^l & \longrightarrow & TB^{2(l-r)} \end{array}$$

commutes. g_1 is the projection. Let

$$F^l = (f^l \wedge \mathrm{id})\colon TB^l \wedge K_{l-r} \longrightarrow Y^l \wedge K_{l-r}$$

and let $\bar{\alpha}^l_r = F^l_* \alpha^l_r$. (5.1) is applicable to $\mathbf{Y}^l$ instead of TB^l so that $\bar{\alpha}^l_r \neq 0$. $(g_2 \wedge \mathrm{id})_* \bar{\alpha}^l_r = \bar{\alpha}^{2(l-r)}_{l-r}$. Let C be defined by

$$C(\alpha, \beta) = (g_2 \wedge \mathrm{id})_* F^l_* \bar{A}(\alpha, \beta) = D(\rho(\alpha), \rho(\beta))\bar{\alpha}^{2(l-r)}_{l-r}.$$

On the other hand $C(\alpha, \beta)$ is the image of $S^{l-2r}(\alpha \wedge \beta)$ under the map

$$S^{l-2r}(TB^m \wedge K_p \wedge TB^n \wedge K_q) \longrightarrow \prod_{2(i+1)>l} \mathbf{K}_i \wedge S^{l-2r}K_l$$

$$\longrightarrow \prod_{2(i+1)>2(l-r)} \mathbf{K}_i \wedge K_{l-r} \longrightarrow \mathbf{Y}^{2(l-r)} \wedge K_{l-r}$$

where the first map is given by $\{d_i\}$ and cup product. $\iota_{l-r} \otimes \iota_{l-r}$ evaluated on the image of $S^{l-2r}(\alpha \wedge \beta)$ is $d(\rho(\alpha) \otimes \rho(\beta))$. Thus it is sufficient to prove

LEMMA (8.6). *Under the map* $Z/2 \approx \pi_{2s}(\Pi_{i \geq s}\mathbf{K}_i \wedge K_s) \longrightarrow \pi_{2s}(\mathbf{Y}^{2s} \wedge K_s)$, 1 *goes to* $\bar{\alpha}^{2s}_s$.

PROOF. Let β be the image of 1. $\rho(\beta) = 0$ since $\Pi\mathbf{K}_i \longrightarrow \mathbf{Y}^{2s}$ is homologically trivial in dimension s. Note $\pi_{2s}(\mathbf{Y}^{2s} \wedge K_s)$ is the homology of K_s with coefficients in $\mathbf{Y}^{2s}$. The coefficient sequence $\Pi\mathbf{K}_i \longrightarrow \mathbf{Y}^{2s} \longrightarrow \mathbf{K}_0 \longrightarrow \Pi\mathbf{K}_i$ gives an exact sequence,

$$\begin{array}{ccccc} H_{2s}(K_s) & \xrightarrow{a} & \sum_{i\geq s} H_{2s-i}(K_s) & \longrightarrow & H_{2s}(K_s; Y^{2s}) \\ & & \wr & & \| \\ & & \pi_{2s}\left(\prod K_i \wedge K_s\right) & \xrightarrow{b} & \pi_{2s}(Y^{2s} \wedge K_s) \end{array}$$

where $a = \Sigma_{i>s} \mathrm{Sq}^i$. $a = 0$ since $\mathrm{Sq}^i\, \iota_s = 0$ for $i > s$. Therefore b is an injection and thus $\beta = \bar{\alpha}_s^{2s}$. The proof of (3.2) is complete.

9. Proof of (3.3) and (4.5).

PROOF OF (3.3). Suppose U_1 and U_2 are two B-Wu orientations on M which are equivalent as B structures on ν_M. As in the proof of (3.2)(v), $U_1, U_2: T(\nu_M) \longrightarrow TB^m$ differ by a map $d: T(\nu) \longrightarrow \Pi_{2q>m} K_q$. Let $D(u)$: $\Sigma_{2p\leq m} H_p(M) \longrightarrow Z_2$ be defined by $\varphi_{U_1}(u) - \varphi_{U_2}(u) = D(u)\alpha_{m-2p}$. We show

$$D(u) = (u \cup d_{m-p})(M) \tag{9.1}$$

as in the proof of (3.2); $(g_2 \wedge \mathrm{id})_* F_*^! \psi_{\iota_i}(u)$ differ, for $i = 1$ and 2, by the image of α_M under

$$\pi_m(T(\nu_M)) \longrightarrow \pi_m(T(\nu_M) \wedge K_p) \xrightarrow{a} \pi_{2(m-p)}\left(\prod_{2q>m} K_q \wedge K_{m-p}\right)$$

$$\longrightarrow \pi_{2(m-p)}(K_{m-p} \wedge K_{m-p}) \longrightarrow \pi_{2(m-p)}(Y^{2(m-p)} \wedge K_{m-p})$$

where $a = (d \wedge u)_*$. $\iota_{m-p} \otimes \iota_{m-p}$ pulls back to $(d_{m-p} \cup u) \cup 1$ in $H^*(T(\nu_M))$. $\iota_{m-p} \otimes \iota_{m-p}$ is one on the generator of $\pi_{2(m-p)}(K_{m-p} \wedge K_{m-p})$. (9.1) now follows from (8.6) and the proof of 3.3 is complete.

PROOF OF (4.5). Recall S^n was B-Wu oriented to form $\bar{S}^n$ as follows: Choose a standard framing of ν_{S^n}, F: $\nu_{S^n} \longrightarrow R^k$ ((S^n, F) is framed cobordant to zero). This gives a B-Wu orientation U_1: $\nu_M \xrightarrow{F} R^k \xrightarrow{i} \zeta_k^n$ where i is the inclusion of a fibre. Let U be the orientation on $\bar{S}^n$ such that U and U_1 differ by the nontrivial map d: $S^n \longrightarrow K_n$. Hence by (3.3),

$$d_U(1) = \varphi_{U_1}(1) + (1 \cup d)(S^n)\alpha_n,$$

$(1 \cup d)(S^n) = 1$ and $\varphi_{U_1}(1) = 0$ since the following diagram is commutative:

$$\begin{array}{ccccc} T(\nu_{S^n}) & \longrightarrow & T(\nu_{S^n}) \wedge S^{n+} & & \\ \big\downarrow U_1 & & \big\downarrow & \searrow & \\ & & & & TB^n \wedge K_0 \\ & & & \nearrow & \\ TB^n & \longrightarrow & TB^n \wedge B^{n+} & & \end{array}$$

and $U_1(\alpha_{S^n}) = 0$. Thus $\varphi_U(1) = \alpha_n$ and (4.5) is proved.

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